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Correlation functions in the Calogero–Sutherland model with open boundaries*

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Abstract. Calogero-Sutherland models of type BC_N are known to be relevant to the physics of one-dimensional quantum impurity effects. Here we represent certain correlation functions of these models in terms of generalized hypergeometric functions. Their asymptotic behaviour supports the predictions of (boundary) conformal field theory for the orthogonality catastrophy and Friedel oscillations.

PACS. 05.70.Jk Critical point phenomena – 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.) – 72.10.Fk Scattering by point defects, dislocations, surfaces, and other imperfections (including Kondo effect)

1 Introduction

One-dimensional models with inverse square interactions have attracted considerable interest in recent years. For the Calogero Sutherland (CS) models [1–3] describing particles moving on a continuous line the many-body ground state wave function is of Jastrow type and excitations can be written as a product of this pair product wave function and certain polynomials in the coordinates. Based on this observation the eigenvalues of the Hamiltonian can be found by means of an asymptotic Bethe Ansatz (ABA) solution [2]. Finite-size scaling analysis of the excitation spectrum and predictions of conformal field theory (CFT) have been used to study the critical behaviour of the CS model, leading to the identification of the universality class of the model with periodic boundary conditions as Luttinger liquid, *i.e.* a Gaussian model with central charge $c = 1$ [4,5]. An interesting property of th CS models is that the compact form of the eigenstates allows for an explicit calculation of certain correlation functions [6–9], thus allowing to compare the asymptotic predictions of CFT to exact expressions derived in a microscopic model.

In addition to the models with periodic boundary conditions there exists a class of CS models lacking translational invariance, in particular the model of BC_N-type which is invariant under the action of the Weyl group of type B_N [3]. Again the ground state can be written in a compact form of Jastrow type, and the spectrum of excitations can be found [10, 11]. An analysis of the finite size corrections to the energies shows that the spectrum acquires contributions due to the "boundaries" of the system and the low-energy critical behaviour is described by a $c = 1$ boundary CFT [12]. Again, the existence of a "simple" expression for the ground state wave function opens the possibility to compare the predictions of boundary CFT for the asymptotic behaviour of correlation functions with exact results. These predictions are of great interest at present due to the possibility to extract observable properties of quantum impurity systems from finite size spectra (see *e.g.* [13–15]).

In the present paper we shall consider the BC_N-type CS model and compute matrix elements which allow to compare the exponents associated with the Anderson's "orthogonality catastrophy", *i.e.* the dependence of the overlap between ground states for *different* boundary conditions on the system size, and the asymptotic behaviour of (Friedel) density oscillations due to the existence of the boundary with the corresponding expressions for the Luttinger liquid [14–20].

First we will give a brief review of the properties of the CS model of BC_N-type as well as the predictions of boundary CFT relevant to the results of this paper. A form of the Hamiltonian in a finite geometry especially

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convenient for our studies is [12]

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda-1)\left(\frac{\pi}{2L}\right)^2 \times \sum_{j<k}\left\{\frac{1}{\sin^2 \frac{\pi}{2L}(q_j-q_k)} + \frac{1}{\sin^2 \frac{\pi}{2L}(q_j+q_k)}\right\} + \mu(\mu-1)\left(\frac{\pi}{2L}\right)^2 \sum_{j=1}^{N} \frac{1}{\sin^2 \frac{\pi}{2L} q_j} + \nu(\nu-1)\left(\frac{\pi}{2L}\right)^2 \sum_{j=1}^{N} \frac{1}{\cos^2 \frac{\pi}{2L} q_j}. \quad (1)$$

Here λ, μ, ν are positive coupling constants. The particles move on the interval $0 \le q_j \le L$. The two particle interaction terms consist of the usual inverse square interaction of particles moving on one half of the circle with circumfence $2L$ plus a term from the interaction of the particle at q_j with the mirror image $-q_k \equiv 2L - q_k$ of the particle at q_k. The last two terms can be regarded as impurity potentials situated at the edges $q = 0$ ($q = L$) of the system with strength given by μ (ν). Eigenvalues and eigenstates of (1) have been determined explicitely [10,11], the many-particle ground state wavefunction is [3]

$$\Psi_0^{(\lambda)}(q_1, \ldots, q_N; \mu, \nu) = \prod_{j<k}\left|\sin\frac{\pi}{2L}(q_j-q_k)\,\sin\frac{\pi}{2L}(q_j+q_k)\right|^{\lambda} \times \prod_{\ell=1}^{N}\left|\sin\frac{\pi}{2L}q_\ell\right|^{\mu}\left|\cos\frac{\pi}{2L}q_\ell\right|^{\nu}. \quad (2)$$

As for the periodic CS models the spectrum can be reproduced exactly by means of the asymptotic Bethe Ansatz method [10]. Expanding the ground state energy in inverse powers of the system size the following finite size scaling form is found [12]

$$E_N^{(0)} = L\epsilon^{(0)} + 2f + \frac{2\pi v_F}{L}\frac{\lambda}{4}(\Delta N_b)^2 - \frac{\pi v_F}{24L}\lambda \quad (3)$$
$$\Delta N_b = \frac{1}{2\lambda}(\mu+\nu-\lambda)$$

where $\epsilon^{(0)}$ and f are the bulk energy density and the boundary energy in the thermodynamic limit for fixed particle density $n = N/L$, respectively. $v_F = 2\pi\lambda n$ is the Fermi velocity of the elementary excitations. Similarly, the energy of an excited state with ΔN additional particles and $N_{ph} > 0$ particle-hole excitations near the Fermi point is, to leading order in $1/L$,

$$E - E_N^{(0)} - \mu_c^{(0)}\Delta N \simeq \frac{2\pi v_F}{L}\left(\frac{\lambda}{4}(\Delta N + \Delta N_b)^2 - \frac{\lambda}{4}(\Delta N_b)^2\right) + \frac{\pi v_F}{L}N_{ph}. \quad (4)$$

Here we have absorbed a term $\mu_c^{(0)}\Delta N$ into the definition of the energy the system with $\mu_c^{(0)} = L\partial\epsilon^{(0)}/\partial N$ being the chemical potential.

The expressions (3, 4) should be compared with the corresponding prediction of CFT for models with free boundary conditions, namely

$$E^{(x)} = L\epsilon^{(0)} + 2f - \frac{\pi v_F}{24L}c + \frac{\pi v_F}{L}x \quad (5)$$

with the Virasoro central charge c appearing in the universal amplitude of the $1/L$-term and the critical exponent x of the operator generating this state. It is well known that the long-range nature of the interactions in CS models gives rise to non-universal $1/L$-contributions to the ground state energy of these systems [5,12]. The λ-dependence in the last term in (3) is believed to be a consequence of this effect, thus yealding an incorrect value for the central charge.

The other contribution of order $1/L$ to the ground state energy is a consequence of the scattering due to the free boundary and the impurity potentials. If one applies boundary CFT to obtain the surface critical exponents controlling the asymptotic behaviour of correlation functions one has to distinguish operators connecting states corresponding to *different* boundary conditions and operators inducing a change of particle number or particle-hole excitations in the ground state corresponding to a given boundary condition [14,15]. For the latter case the phase shift ΔN_b should be absorbed into the the change of the number of particles

$$\tilde{E}_N^{(0)} = E_N^{(0)} - \frac{2\pi v_F}{L}\frac{\lambda}{4}(\Delta N_b)^2,$$
$$\widehat{\Delta N} = \Delta N + \Delta N_b \quad (6)$$

to restore particle hole symmetry of the finite size spectrum (4). The resulting scaling dimension of an operator ϕ corresponding to this situation is

$$x(\phi) = \frac{L}{\pi v_F}\left(E - \tilde{E}_N^{(0)}\right)^2 = \frac{\lambda}{2}\left(\widehat{\Delta N}\right)^2 + N_{ph} \quad (7)$$

with integer $\widehat{\Delta N}$.

To obtain the conformal dimension of boundary condition changing operators finite size energies corresponding to states subject to *different* boundary conditions have to be compared [14,15,17]. Consequently, only one of the two phase shifts ΔN_b, $\Delta N_{b'}$ arising in these expressions can be absorbed into a shift of the particle number as in (6). For the operator $\psi_{bb'}$ connecting the gound states corresponding to different boundary conditions this leads to an operator dimension

$$x(\psi_{bb'}) = \frac{\lambda}{2}(\Delta N_b - \Delta N_{b'})^2. \quad (8)$$

This exponent determines the orthogonality exponent $\langle 0_b|0_{b'}\rangle \propto L^{-x}$ and the related X-ray edge singularity arising from a sudden change of the boundary potential [14,15,17,21].

The asymptotic behaviour of correlation functions in the bulk is still determined by the conformal dimension of the corresponding operator. In fact, an n-point function of the semiinfinite system is subject to different boundary conditions but obeys the same differential equation as the $2n$-point function including the mirror positions in the system without boundary [22]. Hence, the critical exponent for the asymptotic behaviour of the single particle density $\langle\rho(q)\rangle$ due to the presence of the boundary can be obtained from the density density correlation function for the corresponding system without a boundary, namely the periodic CS model. The most dominant term is due to backscattering processes with momentum $\pm 2k_F \equiv 2\pi n$ and decays asymptotically as [5]:

$$\langle\rho(q)\rangle - n \sim \frac{\cos(2k_F q)}{q^{1/\lambda}} \, . \tag{9}$$

2 Overlap integrals

To compute the overlap integral between ground states of (1) corresponding to different values of the boundary field stengths μ, ν we will make use their representation in terms of so called Selberg correlation integrals (see [6] and references therein)

$$S_{n,m}(\lambda_1,\lambda_2,\lambda;x_1,\dots x_m) = \left(\prod_{\ell=1}^{n}\int_0^1 dt_\ell \prod_{p=1}^{m}(t_\ell - x_p)\right) D_{\lambda_1,\lambda_2,\lambda}(t_1,\dots,t_n) \tag{10}$$

with

$$D_{\lambda_1,\lambda_2,\lambda}(t_1,\dots,t_n) = \prod_{\ell=1}^{n} t_\ell^{\lambda_1-1}(1-t_\ell)^{\lambda_2-1} \times \prod_{1\le j<k\le n} |t_k - t_j|^\lambda \, . \tag{11}$$

For $m = 0$ the integrals can be evaluated with result expressed in terms of Gamma-functions

$$S_{n,0}(\lambda_1,\lambda_2,\lambda) = \prod_{j=1}^{n} \frac{\Gamma(1+\lambda j/2)\Gamma(\lambda_1+\lambda(j-1)/2)\Gamma(\lambda_2+\lambda(j-1)/2)}{\Gamma(1+\lambda/2)\Gamma(\lambda_1+\lambda_2+\lambda(n+j-2)/2)} \, . \tag{12}$$

We first note that the normalization integral of (2)

$$\mathcal{N}_{\lambda,\mu,\nu} = \left(\prod_{\ell=1}^{N}\int_0^L dq_\ell\right) |\Psi_0^{(\lambda)}(q_1,\dots,q_N;\mu,\nu)|^2 \tag{13}$$

is of the form (10): Substituting $t_\ell = \frac{1}{2}\left(1-\cos\frac{\pi}{L}q_\ell\right)$ we obtain

$$\begin{aligned}\mathcal{N}_{\lambda,\mu,\nu} &= \left(\prod_{\ell=1}^{N}\frac{L}{2\pi}\int_0^1 dt_\ell\, t_\ell^{\mu-\frac{1}{2}}(1-t_\ell)^{\nu-\frac{1}{2}}\right) \times \prod_{1\le j<k\le n}|t_k-t_j|^{2\lambda} \\ &= \left(\frac{L}{2\pi}\right)^N S_{N,0}\left(\mu+\frac{1}{2},\nu+\frac{1}{2},2\lambda\right) \, .\end{aligned} \tag{14}$$

Now the overlap between two states (2) for different sets of boundary fields (μ,ν) and (μ',ν') can be expressed as the ratio of Selberg correlation integrals

$$|\langle\mu,\nu|\mu',\nu'\rangle|^2 = \frac{(S_{N,0}((\mu+\mu'+1)/2,(\nu+\nu'+1)/2,2\lambda))^2}{S_{N,0}(\mu+1/2,\nu+1/2,2\lambda)S_{N,0}(\mu'+1/2,\nu'+1/2,2\lambda)} \, . \tag{15}$$

Using (12) we obtain

$$\ln|\langle\mu,\nu|\mu',\nu'\rangle| = \frac{1}{2}\sum_{j=0}^{N-1}\Big(f(\mu,\mu',\frac{1}{2}+\lambda j) + f(\nu,\nu',\frac{1}{2}+\lambda j) - f(\mu+\nu,\mu'+\nu',1+\lambda(N+j-1))\Big) \tag{16}$$

with

$$\begin{aligned} f(\mu,\mu',x) &= \ln\frac{\Gamma^2\left(\frac{1}{2}(\mu+\mu')+x\right)}{\Gamma(\mu+x)\,\Gamma(\mu'+x)} \\ &\approx (\mu-\mu')^2\left(-\frac{1}{4x}+\frac{1}{8x^2}(\mu+\mu'-1)+\cdots\right) . \end{aligned} \tag{17}$$

For large system size L and fixed density $\rho = N/L$ the overlap integral (16) is determined by the logarithmic divergence of the first two contributions to the sum at $j = N-1$ giving

$$\ln|\langle\mu,\nu|\mu',\nu'\rangle| \propto \frac{1}{8\lambda}\left((\mu-\mu')^2+(\nu-\nu')^2\right)\ln L \tag{18}$$

which is in perfect agreement with (3) and the prediction (8) of boundary CFT (note that the operator connecting $|\mu,\nu\rangle$ and $|\mu',\nu'\rangle$ in this situation is a product of two boundary changing operators $\psi_{\mu\mu'}$ and $\psi_{\nu\nu'}$ acting at the boundary at $q = 0$ and $q = L$, respectively [17]).

3 Friedel oscillations

We consider now a spatial dependence for single particle density

$$\langle\rho(q)\rangle = \frac{1}{\mathcal{N}_{\lambda,\mu,\nu}} \times \left(\prod_{\ell=2}^{N}\int_0^L dq_\ell\right)|\Psi_0^{(\lambda)}(q,q_2,\dots,q_N;\mu,\nu)|^2 . \tag{19}$$

For integer values λ the function $\langle\rho(q)\rangle$ can be expressed in terms of the Selberg integrals (10)

$$\langle\rho(q)\rangle = \frac{2\pi}{L}x^\mu(1-x)^\nu \times \frac{S_{N-1,2\lambda}\left(\mu+\frac{1}{2},\nu+\frac{1}{2},2\lambda;x_1=\ldots=x_m\right)}{S_{N,0}\left(\mu+\frac{1}{2},\nu+\frac{1}{2},2\lambda\right)} \, , \tag{20}$$

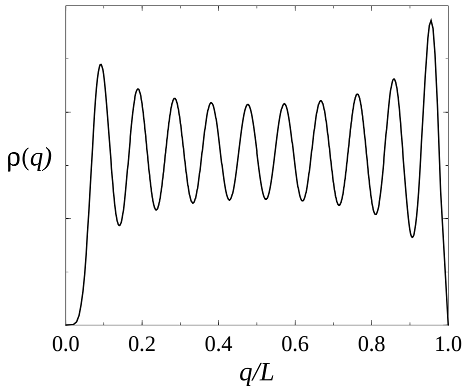

Fig. 1. Single particle density oscillations for a system of $N = 10$ particles with $\lambda = 2$, $\mu = 3$ and $\nu = 1$.

where $m = 2\lambda$ and $x_1 = \ldots = x_m = x = \sin^2(\pi q/2L)$. Equation (20) can be rewritten as

$$\begin{aligned}\langle\rho(q)\rangle = &\frac{2\pi}{L}x^{\mu}(1-x)^{\nu}\\ &\times\frac{S_{N-1,0}(\mu+\frac{1}{2}+2\lambda,\nu+\frac{1}{2},2\lambda)}{S_{N,0}(\mu+\frac{1}{2},\nu+\frac{1}{2},2\lambda)}\\ &\times {}_2F_1^{(\lambda)}(-N+1,\frac{1}{\lambda}(\mu+\nu+m)+N-2,\\ &\times\frac{1}{\lambda}(\mu-\frac{1}{2}+m),x_1=\ldots=x_m),\end{aligned}\tag{21}$$

where ${}_2F_1^{(\lambda)}$ is a generalized hypergeometric function of m variables (see [6]). For finite systems this expression for the single particle density can be evaluated by using the fact that ${}_2F_1^{(\lambda)}$ for equal arguments can be written in tems of Jack symmetric polynomials [23]. In Figure 1 we have plotted (21) in this representation for a system of $N = 10$ particles.

In the thermodynamic limit we derive the asymptotic behavior of $\langle\rho(q)\rangle$ for $1 \ll q \ll N$ using the integral representation [7]

$$\begin{aligned}&{}_2F_1^{(2/\lambda)}(a,\lambda_1+\lambda(m-1)/2,\lambda_1+\lambda_2\\ &+\lambda(m-1),x_1=\ldots=x_m)\\ &=(S_{m,0}(\lambda_1,\lambda_2,\lambda))^{-1}\\ &\times\left(\prod_{\ell=1}^{m}\int_{-\infty}^{0}dt_\ell(1-xt_l)^{-a}\right)D_{\lambda_1,\lambda_2,\lambda}(t_1,\ldots,t_m).\end{aligned}\tag{22}$$

Omitting the x-independent factor we find from (21)

$$\begin{aligned}\langle\rho(q)\rangle \propto\, &x^{\mu}(1-x)^{\nu}\\ &\times\prod_{\ell=1}^{m}\left(\int_0^{\infty}dt_\ell\left\{\frac{(1+xt_\ell)t_\ell}{(1+t_\ell)}\right\}^{\bar{n}}\frac{t_\ell^{(\mu+\nu+1)/\lambda-2}}{(1+t_\ell)^{2+(\nu-\frac{1}{2})/\lambda}}\right)\\ &\times\prod_{j<k}^{m}|t_j-t_k|^{2/\lambda},\end{aligned}\tag{23}$$

where $\bar{n} = N-1$. Near the boundary $q \ll 1$ we obtain

$$\langle\rho(q)\rangle \propto q^{2\mu}.\tag{24}$$

For $1 \ll q \ll N$ we have $x = \sin^2(\pi q/2L) \to g^2/\bar{n}^2$, $g = k_F q/2$. After rescaling $t_l \to \bar{n}t_l$ and using the identity $\lim_{n\to\infty}(1+y/n)^n = \exp y$ we obtain from (23)

$$\begin{aligned}\langle\rho(q)\rangle \propto\, &q^{2\mu}\\ &\times\left(\prod_{\ell=1}^{m}\int_0^{\infty}dt_\ell\exp\left(g^2t_\ell-1/t_\ell\right)t_\ell^{\frac{3}{2\lambda}-4+\frac{\mu}{\lambda}}\right)\\ &\times\prod_{j<k}|t_j-t_k|^{2/\lambda}.\end{aligned}\tag{25}$$

In order to calculate this divergent integral we analytically continue it to imaginary g and obtain after rescaling $t \to t/g$

$$\begin{aligned}\langle\rho(q)\rangle \propto\, &g^{m-1}\\ &\times\left(\prod_{\ell=1}^{m}\int_0^{\infty}dt_\ell\exp^{-g(t_\ell+\frac{1}{t_\ell})}t_\ell^{\frac{3}{2\lambda}-4+\frac{\mu}{\lambda}}\right)\\ &\times\prod_{j<k}|t_j-t_k|^{2/\lambda}.\end{aligned}\tag{26}$$

Now equation (26) can be easy evaluated in the limit $q \gg L/N$ which corresponds to $g \gg 1$. Considering the case $m = 2$ first we find after an integration near the extremum point $t_j = 1$

$$\langle\rho(q)\rangle \propto \frac{\cos 2k_F q}{q^{1/\lambda}}\,\cdot\tag{27}$$

In the general case the main oscillating term is obtained by integration (26) over an region where only two variables are near extremum points. Then the integration over these two variables gives rise to a factor

$$\frac{\cos 2k_F q}{q^{1/\lambda+1}}$$

while the other integrals of type $\prod\int e^{-gt}f(t)$ contribute a factor of order $(1/g)^{m-2}$. Substituting to equation (26) we reproduce the result (27) for any value of the coupling constant λ. Again, this result is in complete agreement with the boundary CFT prediction (9).

4 Free fermionic case

Note that in the free fermionic case $\lambda = 1$ the Friedel oscillations can be investigated in more detail due to a possibility to express the Selberg integral $S_{n,2}$ in terms of Appell's hypergeometric function F_4 [6]

$$\begin{aligned}S_{n,2}(\lambda_1,\lambda_2,\lambda;x_1,x_2) = &(-1)^n S_{n,0}(\lambda_1+1,\lambda_2+1,\lambda)\\ &\times F_4(a,b,c-1,c_2;(1-x_1)(1-x_2)),\end{aligned}\tag{28}$$

where

$$\begin{aligned}&a=-n,\, b=\frac{2}{\lambda}(\lambda_1+\lambda_2+1)+n-1,\\ &c_1=\frac{2\lambda_1}{\lambda},\, c_2=\frac{2\lambda_2}{\lambda}\cdot\end{aligned}\tag{29}$$

Substituting (28) to equation (20) we find

$$\langle\rho(q)\rangle \propto x^{\mu}(1-x)^{\nu} \times F_4(-\bar{n}, \mu+\nu+1+\bar{n}, \mu+\frac{1}{2}, \nu+\frac{1}{2}, x^2, (1-x)^2), \tag{30}$$

where $x = \sin^2(k_F q/2N)$ as before. The Appell function $F_4(\alpha, \beta, \gamma_1, \gamma_2, x, y)$ satisfies the following system of equations

$$x(1-x)Z''_{xx} - y^2 Z''_{yy} - 2xyZ''_{xy} + [\gamma_1 - (\alpha+\beta+1)x]Z'_x - (\alpha+\beta+1)yZ'_y - \alpha\beta Z = 0, \tag{31}$$

$$y(1-y)Z''_{yy} - x^2 Z''_{xx} - 2xyZ''_{xy} + [\gamma_2 - (\alpha+\beta+1)y]Z'_y - (\alpha+\beta+1)xZ'_x - \alpha\beta Z = 0.$$

In the limit $q \ll N$ we obtain after simple manipulations

$$\langle\rho(q)\rangle \propto Z(u,v)|_{u=2q^2, v=0}, \tag{32}$$

where $Z(u,v)$ is the solution of equations

$$Z''_{uv} = \frac{\mu(\mu-1)Z}{(u+v)^2}, \tag{33}$$
$$2uZ''_{uu} - 2vZ''_{vv} - 2(u-v)Z''_{uv} + 3Z'_v + k_F^2 Z = 0.$$

For $u \gg 1$ in the first order variables u and v are separated each from other. We have as a result

$$\langle\rho(q)\rangle \propto n + \frac{1}{q^{1/2}} Z_\delta(2k_F q), \tag{34}$$

where Z_δ is the Bessel function and

$$\delta = \sqrt{1/4 + 8\mu(\mu-1)}.$$

The asymptotic behavior of (34) is in agreement with general result (27).

5 Summary and conclusion

We have studied the asymptotic behaviour of the overlap integral between ground states of the BC_N type Calogero Sutherland model corresponding to different strenghts of the boundary fields and the Friedel oscillations of the single particle density due to the presence of a boundary. Our results are in agreement with those obtained by applying the predictions of boundary conformal field theory to the finite size spectra of these systems. In particular, it has been established that non universal exponents showing a continuous dependence on the values of the boundary fields can arise in the correlation functions of boundary changing operators.

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References

1. F. Calogero, J. Math. Phys. **10**, 2191 and 2197 (1969); *ibid.* **12**, 419 (1971).
2. B. Sutherland, J. Math. Phys. **12**, 246 and 251 (1971); Phys. Rev. A **4**, 2019 (1971); *ibid.* **5**, 1372 (1972).
3. M.A. Olshanetsky, A.M. Perelomov, Phys. Rep. **94**, 313 (1983).
4. F.D.M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).
5. N. Kawakami, S.-K. Yang, Phys. Rev. Lett. **67**, 2493 (1991).
6. P.J. Forrester, Nucl. Phys. B **388**, 671 (1992).
7. P.J. Forrester, J. Math. Phys. **36**, 86 (1995).
8. P.J. Forrester, Mod. Phys. Lett. B **9**, 359 (1995).
9. F. Lesage, V. Pasquier, D. Serban, Nucl. Phys. B **435**, 585 (1995).
10. T. Yamamoto, J. Phys. Soc. Jpn **63**, 1223 (1994).
11. D. Bernard, V. Pasquier, D. Serban, Europhys. Lett. **30**, 301 (1995).
12. T. Yamamoto, N. Kawakami, S.-K. Yang, J. Phys. A **29**, 317 (1996).
13. I. Affleck, in *Correlation effects in low-dimensional electron systems*, Vol. 118 of *Springer Series in Solid-State Sciences*, edited by A. Okiji, N. Kawakami (Springer Verlag, Berlin, 1994), pp. 82–95.
14. I. Affleck, A.W.W. Ludwig, J. Phys. A **27**, 5375 (1994).
15. I. Affleck, Nucl. Phys. B (Proc. Suppl.) **58**, 35 (1997).
16. A.V. Balatsky, S.I. Matveenko, Phys. Rev. B **52**, R8676 (1995).
17. G. Bedürftig, H. Frahm, J. Phys. A **30**, 4139 (1997).
18. R. Egger, H. Grabert, Phys. Rev. Lett. **75**, 3505 (1995).
19. P. Schmitteckert, U. Eckern, Phys. Rev. B **53**, 15397 (1996).
20. F. Lesage, H. Saleur, J. Phys. A **30**, L457 (1997).
21. F.H.L. Eßler, H. Frahm, Phys. Rev. B **56**, 6631 (1997).
22. J.L. Cardy, Nucl. Phys. B **240**, 514 (1984).
23. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. (Clarendon Press, Oxford, 1995).